Almost exponential transverse spectra from power law spectra

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Abstract We point out that exponential shape of transverse spectra can be obtained as the Fourier transform of the limiting distribution of randomly positioned partons with power law spectra given by pQCD, which actually realize Tsallis distributions. Such spectra were used to obtain hadron yields by recombination in relativistic heavy-ion collisions at RHIC energies.

In relativistic heavy-ion collisions exponential transverse momentum distributions has been observed at low to mid transverse momenta, $p_T \approx 1-4$ GeV [1, 2, 3]. This fact was often interpreted as an indication that the source of the particles can be described by a temperature. This view climaxed in the "thermal model" which assumes a situation well described by equilibrium thermodynamics [4].

Regarding experimental data the exponential law is most prominent in the transverse mass spectra which fit the form $\exp((m-m_T)/T)$ with $m_T = \sqrt{m^2+p_T^2}$ at mid-rapidity at SPS and RHIC energies quite well. A tendency of increasing slopes with increasing hadron masses in these spectra is usually interpreted as a sign of flow, on the basis of the relation $T = T_0 + m\langle u^2\rangle$ [6]. A fit to pion, kaon and proton spectra at RHIC gives $T_0 \approx 140$ MeV and $\langle u^2\rangle \approx 0.5-0.6$ at freeze out[7]. A possible interpretation of this fit is a common flow pattern and a *common temperature*, T_0, for all hadrons produced in the heavy-ion collision. This value, converted back to no-flow by using the above formula for pions gives about $T \approx 210$ MeV, which is somewhat above the color deconfinement temperature predicted by lattice QCD calculations [8].

Since there is no known hadronic process which could thermalize on such a short timescale at which this thermal state must have developed, it is tempting to consider *quark level* mechanisms as reason for the observed exponential spectra.

Constituent quark and parton recombination models have been utilized as a picture of hadron formation in relativistic heavy-ion collisions in recent years [9, 10]. This idea roots in the constituent quark model of hadrons originally combining mesons from two, baryons from three quark-like partons. In order to obtain an exponential hadron spectrum by this mechanism, one needs to start with parton spectra which are already exponential in the intermediate p_T regime of $2-4$ GeV. At higher transverse momenta these spectra go over into power laws, as given by pQCD calculations. According to the respective regions of dominance exponential and power law parton distributions are simply added as a phenomenological effort to generate the desired experimental hadron spectra.

In this letter we present a mechanism, by which power law spectra of partons can be combined to give an exponential spectrum. For this process, one needs several partons to recombine to a constituent quark in contrast to the final step of hadron recombination where only two or three such dressed quarks are used. In the following we show that the exponential can be constructed as the *limiting distribution* of cut power laws in the form

$$w(E) = a\left(1+\frac{E}{b}\right)^{-c} \tag{1}$$

with parameters a, b and c fitted to pQCD calculation results as in Refs.[11]. Here we modified somewhat the original formula by using E, the parton energy, instead of the transverse momentum p_T. At zero rapidity it is, however, equal to the transverse mass m_T, and for large transverse momenta to p_T:

$$E = \cosh y\sqrt{p_T^2+m^2} \tag{2}$$

For large transverse momenta $w(E)$ goes over into a power law spectrum: $w \approx ab^c\,|p_T|^{-c}$.

Before turning to the combination of power laws let us briefly recall the familiar case of the central limit theorem [12]. Short tailed probability distributions all lead to a Gaussian probability distribution for the sum of random deviates in the limit of infinitely many independent draws. It is easy to demonstrate this by considering uniform random deviates x_i. Then $w(x_i) = 0.5$ if x_i is in the interval $[-1,1]$ and zero otherwise. We seek the distribution $P_n(x)$ of the scaled sum,

$$x = \sqrt{\frac{3}{n}}\sum_{i=1}^{n} x_i. \tag{3}$$

It is given by the n-fold integral

$$P_n(x) \;=\; \prod_{i=1}^{n}\left(\frac{1}{2}\int_{-1}^{1}dx_i\right)\,\delta\left(x-\sqrt{\frac{3}{n}}\sum_{j=1}^{n}x_j\right). \tag{4}$$

In order to derive the Gaussian in the $n\to\infty$ limit we use its Fourier transform,

$$\tilde{P}_n(k) \;=\; \int_{-\infty}^{+\infty}dx\,e^{ikx}\,P_n(x). \tag{5}$$

The x-integral is easily done, and the remainder factorizes into n equal contributions,

$$\tilde{P}_n(k) \ = \ \left(\frac{\sin(k\sqrt{3/n})}{k\sqrt{3/n}} \right)^n \tag{6}$$

In the large n limit we arrive at

$$\tilde{P}_\infty(k) = \lim_{n\to\infty} \left(1 - \frac{k^2}{2n}\right)^n \ = \ e^{-k^2/2}. \tag{7}$$

The inverse Fourier transformation results in the standard Gaussian

$$P_n(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}. \tag{8}$$

Fig.1 demonstrates that this limit is approached very quickly, already the sum of $n = 3$ uniform random deviates is distributed nearly Gaussian.

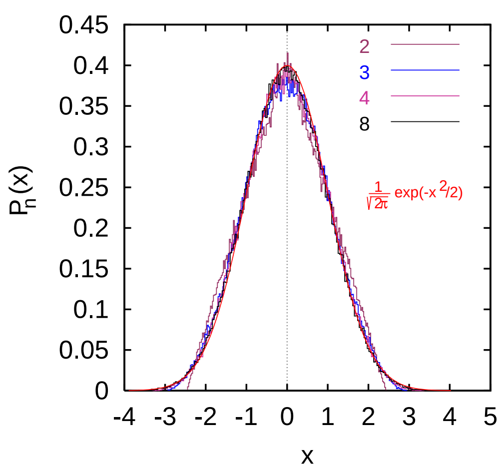

FIG. 1: Comparison of histograms of $m = 200000$ sums of n uniform random deviates in $(-1, 1)$ scaled with $\sqrt{3/n}$ and the limiting Gauss distribution $\frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$.

For the general case one uses an arbitrary distribution, $w(x_i)$ normalized to one, $\int w(x)dx = 1$. The central moments of a distribution are given by higher derivatives of the logarithm of the Fourier transform. For those of the n-fold sum we have

$$c_j(n) = \left(\frac{1}{i}\frac{\partial}{\partial k}\right)^j \log \tilde{P}_n(k)\Big|_{k=0} \tag{9}$$

Since $\log \tilde{P}_n(k) = n \log \tilde{w}(ka_n)$ when looking for the distribution $P_n(x)$ of the scaled sum $x = a_n \sum_i x_i$, we arrive at the following scaling law for the central moments

$$c_j(n) = n a_n^j c_j(1). \tag{10}$$

With a choice of $a_n \propto 1/\sqrt{n}$ one assures that

$$\lim_{n\to\infty} c_j(n) = 0, \qquad \text{for} \quad j > 2. \tag{11}$$

Consequently the limiting distribution, $P_\infty(x)$, is Gaussian.

Now we consider the dressing of constituent quarks with gluons. One may assume that the constituent energy and mass coalesces from n equal, partonic amounts – a kind of additive parton model – but this is not necessary. The total mass in this case would be $M = \sum E_i$, and each parton would carry a fraction of the transverse momentum, $\lambda_i p_T = \mathcal{O}(p_T/n)$. In the following we assume that the spectral shape of the constituent partons (recombining later to hadrons) is proportional to the Fourier transform of the composite wave function. Further we assume, that such a constituent quark (e.g. an up quark) is composed of a bare up quark (the seed) and n gluons (in general, sea partons). Since these partons are almost massless, we consider the center of energy as the proper coordinate for the constituent wave function:

$$x = \frac{\sum E_i x_i}{\sum E_i} = \sum \lambda_i x_i \tag{12}$$

with $\sum \lambda_i = 1$. By equal sharing of the total energy, which is the case whenever the fusion happens at low relative momenta, it is the arithmetic mean (with $1/n$ scaled sum) of random parton coordinates. Its distribution, $P_n(x)$, is interpreted in the present model as the constituent quark wave function squared. Its Fourier transform is proportional to the momentum spectrum and is obtained as n-fold product of individual parton spectra, each carrying a fraction of the total energy. For the sake of simplicity representing all bare parton distribution $\tilde{w}$ by the same distribution, we have, in the general case, n contributions:

$$\tilde{P}_n(k) \ = \ \prod_{i=1}^{n} \tilde{w}(\lambda_i k) \tag{13}$$

with $\sum_1^n \lambda_i = 1$. Considering cut power law parton spectra as given by eq.(1) the n-fold spectrum at $y = 0$ rapidity becomes

$$\tilde{P}_n(p_T) \ = \ \prod_{i=1}^{n} a_i \left(1 + \frac{\lambda_i M_T}{b}\right)^{-c}. \tag{14}$$

Since all λ_i are positive factors and their sum is normalized to one, the limiting distribution for $n \to \infty$ enforces that all $\lambda_i \to 0$ and hence results in an *exponential* distribution, normalized to $\tilde{P}_\infty(0) = 1$:

$$\tilde{P}_\infty(p_T) \ = \ \exp\left(\frac{M - M_T}{b/c}\right). \tag{15}$$

This is the central result of the present letter. It predicts a spectral slope determined by pQCD as being $T = b/c$. From standard fits to the pQCD parton spectra one gets $T \approx 170 - 210$ MeV, depending on the flavor under consideration and details of the fit to the power law distribution from pQCD. This slope has nothing to do with the lattice QCD phase transition.

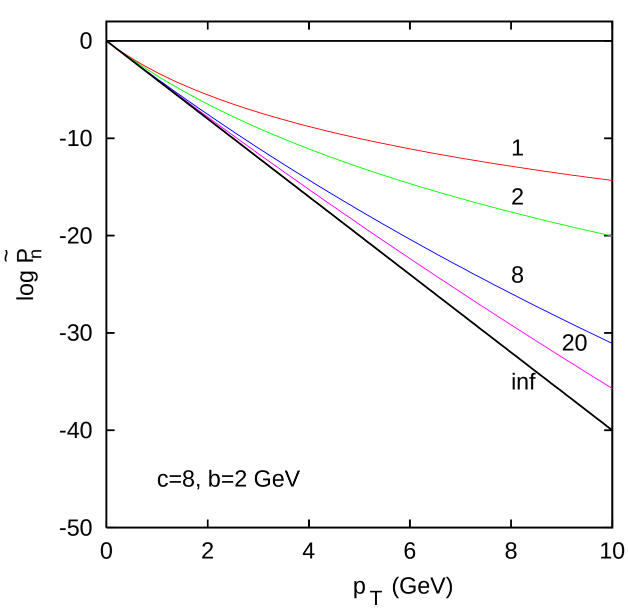

FIG. 2: $\exp(-|p_T|c/b)$, the limiting distribution of cut power law Fourier transforms $(1+|p_T|/b)^{-c}$ and its finite n approximations. The value $n = 20$ is characteristic for the pion.

Our result can be understood by noticing that the cut power law parton distribution of eq.(1) is the canonical Tsallis distribution [13] with a temperature $T = b/c$ and $q = 1 - 1/c$. This distribution maximizes the Tsallis entropy with the usual canonical constraint $\sum_i E_i w(E_i) = const$. The down-scaled product of n such distributions is still of the Tsallis form at the same temperature $T = b/c$ but it belongs to the parameter $q = 1 - 1/(cn)$. In the $n \to \infty$ limit $q \to 1$ and the usual canonical distribution is recovered with the familiar exponential spectrum. By using the rather large values for c usually obtained by fitting pQCD results, namely $c = 8$, one obtains the following series: $q_1 \approx 0.875$, $q_2 \approx 0.94$, $q_3 \approx 0.96$, etc.

It is interesting to recall here another physical application of the Tsallis distribution. Imperfectly thermalized systems, e.g. a finite number of oscillators, also lead to a q value not equal to one. In the case of $(n+1)$ oscillators, however, one arrives at a value bigger than one: $q = 1 + 1/n$[13]. One obtains a distribution reminiscent of the discretized path integral sum,

$$w_i = \frac{1}{Z}\left(1 - \frac{\varepsilon_i}{nT}\right)^n, \tag{16}$$

while primordial parton distributions belong to a q value less than one,

$$w_i = \frac{1}{Z}\left(1 + \frac{\varepsilon_i}{nT}\right)^{-n}. \tag{17}$$

Of course, both distributions approach the Gibbs distribution as $n \to \infty$.

We interpret the high power $c \approx 8 - 9$ in pQCD results as an accidental value simulating an "almost thermal" distribution with an effective $q \approx 0.82 - 0.89$ near to one. This explains why the observed hadron spectra at low and intermediate p_T appear nearly exponential in heavy-ion and many other collisions. Parton recombination to hadrons amplifies this effect leading to $q = 1 - 1/(cn)$ by n-fold recombination. The Tsallis distribution comes very close to the Gibbs distribution. As we shall argue in the following, recombination processes are faster and more effective with increasing phase space volume. Consequently heavy-ion collisions are the best experiments to study strongly interacting matter (QCD) as close to the canonical distribution as possible. (It is a further interesting question whether the primordial parton distribution is described by a smaller power c in e^+e^- collisions than in heavy-ion reactions. This fact could shed a new light onto thermal model interpretation of hadron production in e^+e^- reactions [5]). A remaining question is that how large n has to be for P_n to look exponential. Fig.2 shows this behavior with the parameters $b = 2$ GeV, $c = 8$ (causing $T = 250$ MeV). Here the convergence is slower than in the case of the sum of uniform random deviates approaching a Gaussian. The value $n = 20$ would be characteristic for pion from additive mass estimate $M_\pi/m_q \approx 21.5$ with $m_q = (m_u + m_d)/2$.

Another question is in which proportions the constituent partons, used for recombination to mesons and baryons, may consist of contributions from clusters of perturbative partons of different complexity n. Our suggestion is, that one may not necessarily need to go far in n, i.e. to use already exponential spectra for recombination, in order to have an effective parton distribution close to the usually applied one. In particular, and for the case of simplicity, we discuss the $n = 2$ case here: it improves already the original $n = 1$ distribution a lot in the direction of the assumed exponential.

Fusion of a quark with a gluon and its radiation may be considered as dynamical processes between $n = 1$ and $n = 2$ parton clusters. In order to find an equilibrium ratio of these two type of partons one inspects a rate equation. The detailed balance leads to the familiar proportionality applied in recombination models:

$$f_2(E) = \frac{\gamma}{\Gamma} f_1^2(E/2). \tag{18}$$

The effective spectrum we consider is given by

$$f(E) = A_1 w(E, b, c) + A_2 w^2(E, 2b, 2c). \tag{19}$$

Inspecting the sum of thermal and original power law parton distribution yields used for recombination by Fries. et al. Ref.[10], one concludes that the ratio $A_2/A_1 \approx 0.15$ gives already a surprisingly good approximation in the relevant p_T range from $b \approx 1.5$ GeV to 12

GeV (cf. Fig.3). We note that in the original spectra parton energy loss and a blue shift of the temperature of the thermal part due to radial flow have been included. For our figure we used the same single parton yield, scaled down by a factor 0.87 and added its square at $p_T/2$ with weight 0.13. The corresponding Tsallis "temperature" is $b/c = 190$ MeV.

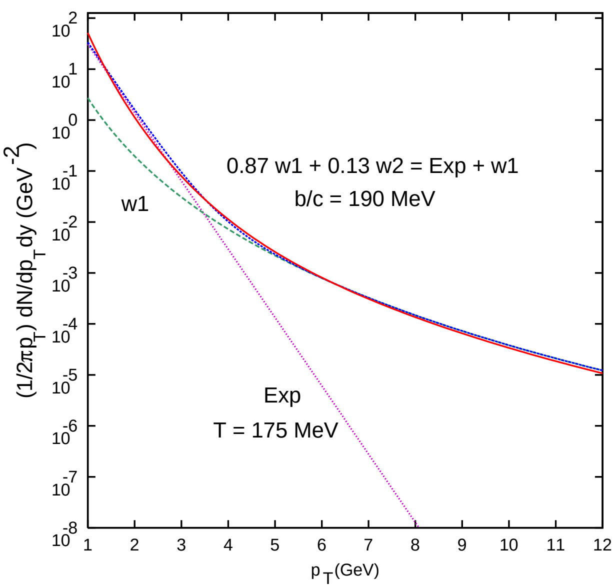

FIG. 3: The thermal and power law parton yield components used by Fries. et al.[10], and the sum of the two contributions is compared to the 13-87% mixture of $n = 1$ and $n = 2$ parton clusters, using the power law distribution only.

For realistic meson and baryon formation a further recombination should be applied for each constituent of a hadron, possibly with different flavor, in the final hadronization process. In a heavy-ion collision event many processes go on in parallel. While some quarks are dressing up to become constituent quarks, some other partons are fragmenting to lower p_T hadrons directly. The hadron p_T spectra then are composed at least from these two components, but maybe also of "partially dressed" quarks if a moderate n in the production of their center of energy from random positioned gluons appear. These may contribute at intermediate p_T to the hadron formation by recombination as well; their yield is, however, reduced compared to the ones behaving nearly exponentially.

The competition between fragmentation and recombination may in general ease the entropy problem of the pure recombination models, where finally one deals with fewer hadrons than partons.

We analyze the entropy problem now for the final hadronization step. As an approximation established by the above discussion, we consider the recombination of exponential spectra from exponential precursors. The entropy can be obtained using the Boltzmann formula (the $q = 1$ case of the more general Tsallis entropy)

$$S = \int (-f \ln f) \frac{d^3p}{(2\pi)^3} V \tag{20}$$

while the number of particles are given by

$$N = \int f \frac{d^3p}{(2\pi)^3} V. \tag{21}$$

The distribution $f(p)$ we consider is proportional to the Fourier transform of the probabilities. Since they were normalized as $\tilde{P}_n(0) = 1$, here we have an unknown factor related to total numbers: $f = a\tilde{P}$. For example for exponential spectra to combine, one gets $f^n(p/n) = a^{n-1} * f(p)$. This leads to the familiar coalescence formula $N_n = C_n N_1^n$ (with C_n being proportional to $(VT^3)^{1-n}$) as well. The n-cluster entropy becomes

$$S_n = (d - n \ln a)\, N_n, \tag{22}$$

due to $\langle p \rangle / T = d$ when considering d-dimensional momentum exponentials. At the same time the relative number of n-clusters is related to the parameter a as $N_n/N_1 = a^{n-1}$.

The sum of all $n > 1$-cluster entropies is then the entropy of the recombined state, S_H, while $S_Q = S_1$ is that of the primordial state. From the phase space integrals one gets

$$S_H = \sum_{n>1} S_n = d \sum_{n>1} N_n - \ln a \sum_{n>1} (nN_n), \tag{23}$$

where the number of quarks redistributed in hadrons occurs as the last term in the expression:

$$\sum_{n>1} nN_n = N_1 = N_Q \tag{24}$$

is the recombination sum rule (counting n quarks in an n-fold cluster). While this rule is quite strict in the recombination of constituents, for dressing up quarks with gluons, of course, such a constraint is not valid. Whenever this sum rule applies, one obtains

$$S_H = S_Q + d \left(N_H - \sum_{n>1} nN_n \right), \tag{25}$$

when expressing $\ln a$ from the $n = 1$ case of eq.(22). Since the recombined sum goes over $n > 1$ indices, and all N_n are by definition positive, $\sum S_n < S_1$ is unavoidable by the recombination of exponential spectra. With other words, after recombination there are fewer hadrons than constituents built into hadrons. The opening of several different flavor channels may ease this problem, but cannot completely solve it. The missing entropy is either generated by i) volume expansion (i.e. the flow must be generated or accelerated by the recombination process)

or by ii) fragmentation, when a single parton produces several hadrons (as it is known from e^+e^- 2-jet events), or by iii) the excitation of internal degrees of freedom of the hadrons.

Based on these principles all hadronic transverse momentum distributions measured at RHIC can be worked out as outlined in this letter. The previously assumed exponential parton spectra now are *derived* in the intermediate p_T range by folding pQCD power law spectra. No assumption of a heat bath, a temperature or thermal equilibrium is required. We present this scenario as an alternative to, and at the same time a possible explanation for, the remarkable phenomenological success of the thermal model.

We do not believe, however, that this result reduces the usefulness of heavy-ion collisions in the search for quark matter. Asymmetric jet quenching and the meson – baryon scaling of the azimuthal flow (v_2) indicate the presence of a collective state with properties of a highly absorptive, perhaps colored, medium. The present explanation of exponential transverse spectra from pQCD and random statistics phenomenology offers an alternative to the ad hoc assumption of a thermal state. In fact, the independence of dressing gluons when combined to a constituent parton is an ingredient of getting the exponential p_T distribution. This condition is probably satisfied only in heavy-ion collisions. Our model also calls attention to the fact that "statistical" is not always synonymous with equilibrium Gibbs thermodynamics. Certain dynamical phenomena, like e.g. self organizing criticality and – what we have just discussed – limiting parton distributions, belong to a more general framework of statistical physics.

The exponential m_T-spectra in fact approach at low p_T a Gaussian, $\exp(-p_T^2/(2mT))$, the textbook case of a thermal distribution. It is interesting to note that several recent articles consider a Gauss distributed intrinsic parton p_T in the generalized parton distribution functions (GPD-s) [14]. The Tsallis distribution predicts $\frac{1}{2}\langle p_T^2\rangle = m_T(b+m_T)$ leading to $\langle p_T^2\rangle \approx 1$ GeV^2, if using a constituent mass of $m_T = 0.3$ GeV and the cut-off $b = 1.5$ GeV, in good agreement with recent theoretical analysis of high energy pp data.

We have seen, however, that the parameter T in this distribution is not related to a heat bath or another source of statistical fluctuations. It is combined from two pQCD parameters, from a cut-off b and a power c as being $T = b/c$. The "heating agent" in a general sense here is the effective number of independent dressing gluons per constituent, n, which should lead to a Tsallis distribution with $q = 1 - 1/(cn)$ quite close to an exponential at low p_T. If the transverse positions of the dressing partons are averaged as independent random variables, the distribution of their center of energy will have a Fourier spectrum, which is nearly exponential up to $p_T \approx 3$ GeV.

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